Asymptotic analysis for the strip problem related to a parabolic third - order operator

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Abstract—Aim of this paper is the qualitative analysis of a boundary value problem for a third order non linear parabolic equation which describes several dissipative models. When the source term is linear, the problem is explictly solved by means of a Fourier series with properties of rapid convergence. In the non linear case, appropriate estimates of this series allow to deduce the asymptotic behaviour of the solution.

Keywords— Partial differential equations, Viscoelastic models, Superconductivity, Asymptotic analysis.

1. INTRODUCTION

The evolution of several dissipative phenomena can be described by the non linear equation

$$\mathcal{L}_\varepsilon u = (\partial_{xx}(\varepsilon\partial_t + c^2) - \partial_t(\partial_t + a))u = F(x,t,u), \tag{1.1}$$

where ε, a, c are positive constants and $F = F(x,t,u)$ is a prefixed function. For instance, the equation (1.1) governs sound propagation in viscous gases as reported in the classical literature [1], motions of viscoelastic fluids or solids [2], or motions of heat condution at low temperature (see, e.g., [3] and refecence therein). Moreover, in superconductivity, (1.1) models the flux dynamics in the Josephson junction. In this case, let $u = u(x,t)$ be the phase difference of the wave functions related to the two superconductors, and let γ be the normalized current bias; when $F = \sin u - \gamma$, equation (1.1) is the perturbed Sine - Gordon equation (PSGE). Terms εu_{xxt} and au_t characterize the dissipative normal electron current flow along and across the junction and represent the *perturbations* with respect to the classic Sine Gordon equation [4]. Moreover, the value's range for a and ε depends on the real junction. In fact, there are cases with $0 < a, \varepsilon < 1$ [5] and, when the resistence of the junction is so low to short completely the capacitance, the case a large with respect to 1 arises [6]. As it is well known, up to today, Josephson junctions have found a lot of applications in many fields and lately new high-T_C materials imply an increasing evolution of the superconducting technologies [7].

If ℓ is the normalized length of the junction, boundary conditions for $x = 0$ and $x = \ell$ specify the applied external magnetic field [8, 9]. Then, if

$$\Omega = \{(x,t) : 0 < x < \ell, \quad 0 < t \leq T\},$$

the strip problem for (1.1) can be stated as follows:

$$\begin{cases} \partial_{xx}(\varepsilon u_t + c^2 u) - \partial_t(u_t + au) = F(x,t,u), \quad (x,t) \in \Omega, \\ u(x,0) = g_0(x), \quad u_t(x,0) = g_1(x), \quad x \in [0,\ell], \\ u(0,t) = 0, \quad u(\ell,t) = 0, \quad 0 < t \leq T. \end{cases} \tag{1.2}$$

The linear case($F = f(x,t)$) has already solved in [10, 11] by means of convolutions of Bessel functions. However, it seems difficult to deduce an exhaustive asymptotic analysis by means of this solution.

In this paper, both linear and non linear problems are analyzed and the Green function of the linear case is determined by Fourier series:

$$G(x,t,\xi) = \frac{2}{l}\sum_{n=1}^{\infty} H_n(t) \quad \sin\gamma_n\xi \quad \sin\gamma_n x, \tag{1.3}$$

where

$$H_n(t) = \frac{1}{\omega_n} e^{-h_n t}\sinh(\omega_n t), \quad \gamma_n = \frac{n\pi}{l}, \tag{1.4}$$

$$b_n = c\gamma_n, \quad h_n = \frac{1}{2}(a+\varepsilon\gamma_n^2), \quad \omega_n = \sqrt{h_n^2 - b_n^2}.$$

This series is characterized by properties of rapid convergence and allows to establish an exponential decrease of the solution in the linear case. Moreover, the non linear problem is reduced to an integral equation with kernel G and, by means of suitable properties of G, the asymptotic analysis of the solutions is achieved.

2. PROPERTIES OF THE GREEN FUNCTION

Firstly, let us prove that function G defined by series (1.3) decreases at least with n^{-2} and that it is exponentially vanishing as $t \to \infty$. In fact, if we put:

$$p = \frac{c^2}{\varepsilon + a(\ell/\pi)^2}, \qquad q = \frac{a+\varepsilon(\pi/\ell)^2}{2}, \qquad \beta \equiv \min(p,q), \tag{2.1}$$

the following lemma holds:

Lemma 2.1- *The function $G(x,\xi,t)$ defined in* (1.3) *and all its time derivatives are continuous functions in Ω, $\forall\ a,\varepsilon,c^2 \in \Re^+$. Moreover, everywhere in Ω, it results:*

$$|G(x,\xi,t)| \le M\, e^{-\beta t}, \qquad |\frac{\partial^j G}{\partial t^j}| \le N_j\, e^{-\beta t}, \qquad j \in \mathsf{N} \tag{2.2}$$

where M, N_j are constants depending on a,ε,c^2.

Proof. When $c^2 = a\varepsilon$, the operator L_ε can be reduced to the wave operator. If $c^2 < a\varepsilon$, for all $n \ge 1$, one has $h_n > b_n$. So, if k is an arbitrary constant such that $c^2/a\varepsilon < k < 1$, Cauchy inequality assures that:

$$\frac{b_n}{\sqrt{k}} \le \frac{1}{2}\ (\varepsilon\gamma_n^2 + a) = h_n. \tag{2.3}$$

Then, if $X_n = (b_n/h_n)^2$, it results:

$$(1-k)^{\frac{1}{2}} \le \frac{\omega_n}{h_n} = (1-X_n)^{\frac{1}{2}} < 1 - X_n/2. \tag{2.4}$$

So, considering that:

$$\frac{b_n^2}{2h_n} \ge p, \qquad h_n > n^2\,(q - a/2), \qquad \forall n \ \ge 1, \tag{2.5}$$

$e^{-t(h_n-\omega_n)} \le e^{-pt}$ and $\omega_n \le \frac{(1-k)^{-\frac{1}{2}}}{q-a/2}\,\frac{1}{n^2}$ hold. As consequence, $(2.2)_1$ follows.

As for $(2.2)_2$, we observe that $h_n - \omega_n \le \frac{2c^2}{\varepsilon}$, $\forall n \ge 1$, and by means of standard computations $(2.2)_2$ is deduced.

It may be similarly proved that estimates (2.2) hold also when $c^2 > a\varepsilon$. ∎

As for the x-differentiation of Fourier's series like (1.3), attention must be given to convergence problems. Therefore, we consider x-derivatives of the operator $(\varepsilon\partial_t + c^2)G$ instead of G and G_t.

Lemma 2.2.- *For all $a, \varepsilon, c^2 \in \Re^+$, the function $G(x,\xi,t)$ defined in (1.3), is such that:*

$$|\varepsilon\, G_t + c^2 G| \le A_o\ e^{-\beta t}, \qquad |\partial_{xx}(\varepsilon\, G_t\ + c^2\, G)| \le A_1\ \ e^{-\beta t}, \tag{2.6}$$

where A_i $(i = 0, 1)$ are constants depending on a, ε, c^2.

Proof. As for the hyperbolic terms in H_n, it results:

$$\varepsilon \dot{H}_n + c^2 H_n = \frac{1}{2\omega_n} e^{-h_n t}\{(c^2 + \varepsilon\varphi_n)e^{\omega_n t} - [c^2 - \varepsilon(h_n + \omega_n)]e^{-\omega_n t}\}, \tag{2.7}$$

with $\varphi_n = h_n(-1 + \sqrt{1 - X_n})$ and $X_n = (\frac{b_n}{h_n}) < 1$.
Further, by means of Taylor's formula it's possible to prove that:

$$|c^2 + \varepsilon\ \varphi_n|\ \ \le \frac{1}{n^2}\ (5\alpha^2/4 + \alpha_1/8), \qquad (\alpha, \alpha_1\ \ = \text{const}). \tag{2.8}$$

Estimates of Lemma 2.1 together with (2.8) show that the terms of the series related to the operator $(2.6)_1$ decrease at least as n^{-4}. So, it can be differentiated term by terms with respect to x and estimate$(2.6)_2$ holds. ∎

We mean as solution of the equation $\mathcal{L}_\varepsilon v = 0$ a continuous function $v(x,t)$ which has continuous the derivatives v_t, v_{tt}, $\partial_{xx}(\varepsilon v_t + c^2 v)$ and these derivatives verify the equation. Therefore, the following theorem holds:

Theorem 2.1- *The function $G = G(x,t)$ defined in* (1.3)*is a solution of the equation*

$$\mathcal{L}_\varepsilon G\ = \partial_{xx}(\varepsilon G_t + c^2 G) - \partial_t(G_t + aG) = 0. \tag{2.9}$$

∎

To obtain the explicit solution of (1.2), properties for the convolution of function G with data have to be analized.
For this, let $g(x)$ be a continuous function on $(0, \ell)$ and consider:

$$u_g(x,t) = \int_0^\ell g(\xi)\ \ G(x,\xi,t)\ \ d\xi, \qquad u_g^*(x,t) = (\partial_t + a - \varepsilon\partial_{xx})u_g(x,t). \tag{2.10}$$

Then one has:

Lemma 2.3- *If $g(x)$ is a $C^1[0,\ell]$, then the function u_g is a solution of the equation $\mathcal{L}_\varepsilon = 0$ such that*:

$$\lim_{t\to 0} u_g(x,t) = 0, \qquad \lim_{t\to 0} \partial_t u_g(x,t) = g(x), \tag{2.11}$$

uniformly for all $x \in [0,\ell]$.

Proof. Lemma (2.2)-(2.3) and continuity of g assure that function $(2.10)_1$ and partial derivatives required by the solution converge absolutely for all $(x,t) \in \Omega$. Hence, since Theorem 2.1, $\mathcal{L}_\varepsilon u_g = 0$. Besides, hypotheses on the function g and (2.2), imply $(2.11)_1$. Further, one has:

$$G_t = -\frac{2}{\pi}\ \frac{\partial}{\partial\xi}\ \sum_{n=1}^{\infty}\ \dot{H}_n(t)\frac{\cos\gamma_n\xi}{n}\sin\gamma_n x, \tag{2.12}$$

and hence:

$$\partial_t u_g = -\frac{2}{\pi}\sum_{n=1}^{\infty}\ \frac{\dot{H}_n(t)}{n}\ \ [g(\xi)\cos\gamma_n\xi\ ]_0^\ell\ \ \sin\gamma_n x \tag{2.13}$$

$$+\frac{2}{\pi}\int_0^\ell \sum_{n=1}^{\infty} \quad \dot{H}_n(t)\, g^{'}(\xi)\, \frac{\cos\gamma_n\xi}{n}\, \sin\gamma_n x \; d\xi.$$

So, if $\eta(x)$ is the Heaviside function, from (2.13), one obtains:

$$\lim_{t\to 0}\partial_t u_g = \frac{x}{\ell}[g(\ell)-g(0)] + g(0) - \int_0^\ell g^{'}(\xi)[\eta(\xi-x)+\frac{x}{\ell}-1]d\xi = g(x). \tag{2.14}$$

∎

Lemma 2.4- *If $g(x)\in C^3[0,\ell]$ with $g^{(i)}(0)=g^{(i)}(\ell)=0 \;\; (i=1,2,3)$, then the function u_g^* defined in $(2.10)_2$ is a solution of the equation $\mathcal{L}_\varepsilon = 0$ such that*:

$$\lim_{t\to 0} u_g^*(x,t) = g(x), \qquad \lim_{t\to 0}\partial_t u_g^*(x,t) = 0, \tag{2.15}$$

uniformly for all $x\in[0,\ell]$.

Proof. Hypoteses on $g(x)$ assure that:

$$\partial_{xx}u_g(x,t) = \int_0^\ell g^{''}(\xi) \;\; G(x,\xi,t) \;\; d\xi = u_{g^{''}}(x,t). \tag{2.16}$$

So, by Lemma 2.3 and expression (2.16), equation $\mathcal{L}_\varepsilon u_g^* = 0$ is verified. Moreover, since $\partial_t u_g^* = c^2 u_{g''}$ and $(2.11)_1$, $(2.15)_2$ holds. Finally, owing to (2.11)-(2.16), one has $(2.15)_1$, too. ∎

3. EXPLICIT SOLUTION OF THE LINEAR PROBLEM

Consider the linear problem. When $F\equiv 0$, by Lemma 2.3 and 2.4, one has:

Theorem 3.1- *If the initial data $g_1(x)$, and $g_0(x)$ verify hypotheses of Lemma* 2.3 and 2.4, *respectively, then the homogeneous problem* (1.2) *admittes the following solution*:

$$u_0(x,t) = u_{g_1} + (\partial_t + a - \varepsilon\partial_{xx})u_{g_0}. \tag{3.1}$$

∎

As for the linear non-homogeneous problem, ($F\equiv f(x,t)$), let:

$$u_f(x,t) \;\; = -\int_0^t d\tau \int_0^\ell f(\xi,\tau) \;\; G(x,\xi,t-\tau) \;\; d\xi. \tag{3.2}$$

By means of standard computations, one can verify that the function (3.2) satisfies (1.2) with $g_0 = g_1 = 0$. As consequence, one has:

Theorem 3.2- *If f and f_x are continuous functions in Ω, then the function $u = \;\; u_0 + u_f$ represents a solution of the linear non-homogeneous strip problem.*

Proof. Considering that:

$$\lim_{\tau\to t}\int_0^\ell f(\xi,\tau)G_t(x,\xi,t-\tau)d\xi = f(x,t), \tag{3.3}$$

Theorem 2.1 assure that $\mathcal{L}_\varepsilon u_f = f(x,t)$.

Moreover, since:

$$|u_f| \le B_1(1-e^{-\beta t}); \quad |\partial_t u_f| \le B_2(1-e^{-\beta t}), \quad (B_1, B_2 \; = \text{const.}), \tag{3.4}$$

initial homogeneous conditions are satisfy. ∎

Uniqueness is an obviously consequence of the energy-method (see, for example, [12]). So, the following theorem holds: :

Theorem 3.3- *When the source term* $f(x,t)$ *satisfies Theorem* 3.2, *and the initial data* (g_0, g_1) *satisfy Theorem* 3.1, *then the function*

$$u(x,t) = u_{g_1} + (\partial_t + a - \varepsilon\partial_{xx})u_{g_0} + u_f \tag{3.5}$$

is the unique solution of the linear non-homogeneous strip problem with $F = f(x,t)$. ∎

Consider, now, the *non-linear* problem (1.2), and observe that (3.5) implies:

$$u(x,t) = \int_0^\ell g_1(\xi)G(x,\xi,t)d\xi + (\partial_t + a - \varepsilon\partial_{xx})\int_0^\ell g_0(\xi)G(x,\xi,t)d\xi \tag{3.6}$$

$$-\int_0^t d\tau \int_0^\ell G(x,\xi,t-\tau)\,F(\xi,\tau,u(\xi,\tau))d\xi.$$

which represents an integral equation for u(x,t).

4. ASYMPTOTIC PROPERTIES

Obviously, asymptotic properties depend on the shape of the source term F. Therefore, when $F = 0$, Lemma 3.1 assures that the solution *exponentially vanishes* when t tends to infinity.

When $F = f(x,t)$ one has:

Theorem 4.1- *When the source term* $f(x,t)$ *satisfies condition*:

$$|\,f(x,t)\,| \,\leq\, h\,\frac{1}{(k+t)^{1+\alpha}} \qquad\qquad (h, k, \alpha = \text{const} > 0), \tag{4.1}$$

then solution (3.5) *is vanishing as* $t \to \infty$, *at least as* $t^{-\alpha}$.

Moreover, when

$$|\,f(x,t)\,| \,\leq\, C\,e^{-\delta t}, \qquad\qquad (C, \delta = \text{const} > 0), \tag{4.2}$$

one has:

$$|\,u(x,t)\,| \,\leq\, k\,e^{-\delta^* t}, \qquad \delta^* = min\{\beta, \delta\}, \quad k = \text{const}. \tag{4.3}$$

Proof. Since $u_{g_i} = O(e^{-\beta t})$ (i=0,1), the proof of the theorem follows directly from the assumptions on $f(x,t)$. ∎

As for the *non linear* problem, previous estimates can be applied. For instance, if we refer to PSGE, it results:

$$|F(x,t,u)| = |\sin u + \gamma| \;\leq \gamma_1, \qquad \gamma_1 = \text{const}. \tag{4.4}$$

Then, the solution of the related non linear strip problem is *bounded for all t.*

Results concerning the exponential decay can be obtained when stronger assumptions on the decreasing properties of the source F(x,t,u(x,t)) are imposed. For instance, according to [13], when

$$|F(x,t,u(x,t))| \leq \text{const.}\;\; e^{-\mu t} \quad (\mu > 0), \tag{4.5}$$

then, the solution u(x,t) of the *non linear* problem (1.2) vanishes exponentially, due to properties (2.2) of G.

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